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1998 J. Phys.: Condens. Matter 10 3089

(http://iopscience.iop.org/0953-8984/10/13/024)

J. Phys.: Condens. Matter **10** (1998) 3089–3092. Printed in the UK PII: S0953-8984(98)84127-1

COMMENT

Comment on 'Boson–fermion model beyond the mean-field approximation'

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Received 13 May 1997

Abstract. In a recent paper, it has been suggested that there is no Cooper pairing in boson–fermion models of superconductivity. This result is based on an inconsistent approximation which violates an exact identity. Quite generally, the divergence of the fermion t-matrix (the Thouless criterion) is accompanied by the condensation of a boson mode.

In a recent paper [1], Alexandrov has argued that there is no Cooper pairing in boson–fermion models [2, 3]. It was suggested that the Thouless criterion for the onset of long-range order is inconsistent with the requirement that the physical energy of a boson be non-negative. Alexandrov's initial argument was based on two approximations.

(A) The fermion t-matrix is approximated by a ladder series.
(B) The boson self-energy is approximated by a single two-fermion bubble.

In a subsequent publication [4], the approximate boson self-energy was supplemented with repeated scattering of intermediate fermions on each other via the Coulomb potential. We will refer to this refined choice as (B′). It turned out [4] that approximations (A) and(B′) no longer contradict each other. Moreover, together they lead to a remarkable conclusion.

(C) *The fermion t-matrix diverges at the boson condensation temperature.*

In other words, the BCS and Bose–Einstein condensations are one and the same thing in this model.

As the erroneous earlier result [1] demonstrates, to prove this conjecture by using an approximate method is no easy task: two separate approximations must be made in a mutually consistent way. Moreover, there is no way to check whether the approximate result will hold if higher-order corrections are included. In this comment, we shall point out an *exact* identity relating the boson propagator to the fermion t-matrix. The purpose of this is twofold. First, this identity establishes the exact equivalence of BCS and Bose condensations in a wide class of boson–fermion models. Second, it provides an easy way to check self-consistency of a given approximation. For illustration purposes, we will show that selecting approximation (A) for the fermion t-matrix fixes the form of the boson self-energy, and that the latter is given precisely by (B′), and not (B).

Let us briefly summarize some of the results obtained by Alexandrov. Although equation (7) in [1], which determines T_c, was derived from the linearized gap equation,

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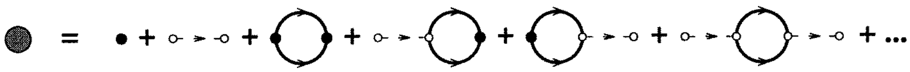

Figure 1. The series for the fermion t-matrix (shaded circle) in Alexandrov's approximation (A), up to second order in $V \equiv V_c + v^2 D_0(\boldsymbol{q}, \Omega_n)$. Open circle: boson–fermion coupling v. Filled circle: on-site repulsion V_c. Thick solid line: dressed fermion propagator $G(\boldsymbol{k}, \omega_n)$. Thin broken line: bare boson propagator $D_0(\boldsymbol{q}, \Omega_n)$.

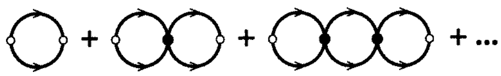

Figure 2. Boson self-energy in the ladder approximation (B′). Approximation (B) uses the first diagram only.

an alternative (but fully equivalent) derivation uses the divergence of the fermion t-matrix at zero energy and momentum as a criterion for the transition [5]. Alexandrov adopts an effective short-range fermion interaction in the form $V = v^2 D_0(0, 0) + V_c$ (the first two diagrams in figure 1), where $D_0(\boldsymbol{q}, \Omega_n)$ is the free boson propagator, v is the boson–fermion coupling and V_c is a short-range fermion repulsion. In the ladder approximation, one obtains the usual result for the t-matrix:

$$T(0, 0) = \frac{V}{1 - V\mathcal{B}} \tag{1}$$

where, in Alexandrov's notation, $\mathcal{B} = -N^{-1} \int \mathrm{d}p G(p) G(-p)$ is a shorthand for a bubble with two fermion lines. The divergence of the t-matrix then requires that

$$1 - \left[v^2 D_0(0, 0) + V_c\right]\mathcal{B} = 0 \tag{2}$$

which is precisely equation (7) of Alexandrov [1].

Consider now the condition for Bose condensation using approximation (B′) for the boson self-energy (figure 2). The boson propagator at $\boldsymbol{q} = \boldsymbol{0}$ and $\Omega_n = 0$ is given by the equation

$$D^{-1}(0, 0) = D_0^{-1}(0, 0) - \frac{v^2 \mathcal{B}}{1 - V_c \mathcal{B}}. \tag{3}$$

At the condensation temperature, bosons with zero momentum have zero energy, i.e., $D(0, 0) = \infty$, and we recover equation (2). Thus, (A) and (B′) are mutually consistent.

We now derive the exact identity shown in figure 3. It indicates that the divergence of $T(0, 0)$ *must* be accompanied by that of $D(0, 0)$. The proof is based on equations of motion for a non-self-interacting boson field $\phi(x, t)$ with a Hamiltonian

$$H_B = \int \mathrm{d}x \phi^\dagger(x, t) \mathcal{H}_B(x) \phi(x, t) \tag{4}$$

where $\mathcal{H}_B$ is a linear operator acting on x. The boson field is coupled to a fermion field $\varphi_\sigma(x, t)$ by the interaction Hamiltonian

$$H_{BF} = v \int \mathrm{d}z \left[\phi^\dagger(x, t) \varphi_\uparrow(x, t) \varphi_\downarrow(x, t) + \mathrm{HC}\right]. \tag{5}$$

The remaining part of the Hamiltonian H_F, which contains fermionic variables only, is unrestricted. In what follows, x denotes spatial coordinates and the (complex) time t varies along a straight line between 0 and $\tau = -\mathrm{i}\hbar / k_B T$.

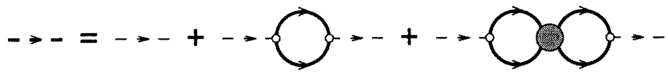

Figure 3. An identity for the exact boson propagator (thick dashed line). Thin dashed line: free boson propagator.

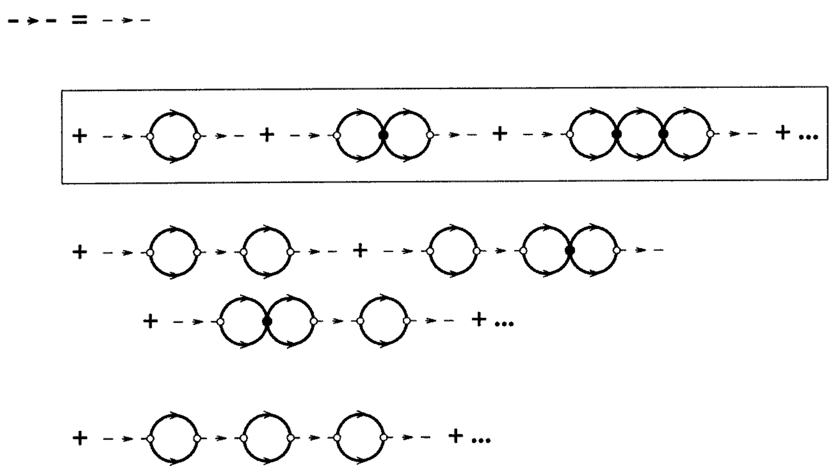

Figure 4. The boson propagator obtained in a self-consistent approximation (by combining figures 1 and 3). The diagrams are grouped according to the number of free boson lines. The boxed set of graphs (with external boson lines amputated) gives the boson self-energy).

Introduce time-ordered boson (D), two-fermion (G_2) and mixed (M) propagators

$$D(x,\ t;\ x',\ t') = -\mathrm{i}\langle T[\phi(x,\ t)\phi^{\dagger}(x',\ t')]\rangle \tag{6}$$

$$M(x,\ y,\ t;\ x',\ t') = -\mathrm{i}\langle[\varphi_{\uparrow}(x,\ t)\varphi_{\downarrow}(y,\ t)\phi^{\dagger}(x',\ t')]\rangle \tag{7}$$

$$G_2(x,\ y,\ t'\,x',\ y',\ t') = -\mathrm{i}\langle T[\varphi_{\uparrow}(x,\ t)\varphi_{\downarrow}(y,\ t)\varphi_{\downarrow}^{\dagger}(y',\ t')\varphi_{\uparrow}^{\dagger}(x',\ y')]\rangle. \tag{8}$$

By using equations of motions for ϕ and $\phi^{\dagger}$, we obtain

$$[\mathrm{i}\hbar\partial/\partial t - \mathcal{H}_B(x)]D(x,\ t;\ x',\ t') = \delta(x-x')\delta(t-t') + vM(x,\ x,\ t;\ x',\ t') \tag{9}$$

$$[-\mathrm{i}\hbar\partial/\partial t' - \mathcal{H}_B(x')]M(x,\ x,\ t;\ x',\ t') = vG_2(x,\ x,\ t;\ x',\ x',\ t'). \tag{10}$$

These differential relations can be integrated with the aid of the free ($v = 0$) boson propagator $D_0(x,\ t;\ x',\ t')$. The following identity is then obtained for the corresponding Fourier coefficients:

$$D(x;\ x'|n) = D_0(x;\ x'|n) + v^2 \int \mathrm{d}x_1 \int \mathrm{d}x_2 D_0(x;\ x_1|n) G_2(x_1,\ x_1;\ x_2,\ x_2|n) D_0(x_2;\ x'|n). \tag{11}$$

$D(x;\ x'|n)$ is defined in a standard way,

$$D(x,\ t'\,x',\ t') = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} D(x;\ x'|n)\exp[-\mathrm{i}\Omega_n(t-t')] \tag{12}$$

where $\Omega_n = 2\pi n/\tau$. Finally, we isolate the disconnected part of G_2 (GG) and express the connected part in terms of the proper vertex ($GGTGG$), as shown in figure 3.

We are now in position to justify choice (B$'$) by Feynman diagrams for the approximate boson self-energy (figure 2). To this end, we can determine the dressed boson propagator directly from the identity of figure 3, by using approximation (A) for the t-matrix (figure 1) as the input. In this way, the exact result (11) is built into the approximate theory from the start. The resulting boson propagator is shown in figure 4. Higher-order graphs replaced with the dots are generated by similarly omitted graphs of figure 1. Diagrams without internal free boson lines determine the boson self-energy. By inspection, the latter is the same as in figure 2.

Note added in proof. After this comment was submitted for publication the authors were made aware of similar results [6].

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